PREDICTION OF THERMAL BOUNDARY LAYER IN THE ENTRY REGION OF A TUBE BY ORTHOGONAL COLLOCATION

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Abstract Energy equation has been solved in the entrance section of a tube for fully developed velocity profile condition for laminar flow of incompressible fluids. Two different boundary conditions, e.g., constant wall temperature and constant wall heat flux have been used.The governing partial differential equation is converted into a set of ordinary differential equations using orthogonal collocation technique. BDFSH- a modified Gear method, which is capable of solving systems of parabolic, hyperbolic, and stiff differential equations coupled with algebraic equations, is used to solve the resulting set of ordinary differential equations. Numerical solutions have been obtained at the very entrance region of the tube. At this low range of axial distance literature values are not available. For relatively higher axial distances analytical solution is available and the results from present study compare very closely with that. At the very entrance, Nusselt number is quite high and decreases quickly with increasing axial distance to its analytical asymptotic value. The same partial differential equation is also solved by using usual finite difference technique. Orthogonal collocation method is found to be at least 5 times better than the later method in both computational time and number of grid points.

Keywords: Entry region, thermal boundary layer, orthogonal collocation.

INTRODUCTION

Investigation of fluid flow in the entrance region of a tube is of considerable practical significance and not surprisingly, there exist a large number of references in the literature on this topic, specially for incompressible laminar flow [Schiller, 1922; Graetz, 1885]. It has attracted much attention by its intrinsic practical importance [Longwell, 1957]. In this respect, the main objective of this work was to determine the temperature profile at the very entrance (say at $\zeta<10^{-5}$ where ζ=dimensionless axial distance, z/R/(Re.Pr)) section of a tube since solutions in this region in not available in the literature. Due to very stiff nature of the problem it is very difficult to obtain solution in this range. Kays [Kays, 1955] was able to obtain solutions at ζ=0.0002 which gave too high values for the local Nusselt number compared to analytical solution obtained by Lipkis [Lipkis, 1954]. In this work it has been found that the numerical solutions are in excellent agreement with the analytical results where they are available.

The governing partial differential equation is converted into a set of ordinary differential equations using orthogonal collocation technique. Two different boundary conditions, e.g., constant wall temperature and constant wall heat flux have been used.

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FORMULATION

When an incompressible Newtonian fluid with constant viscosity and thermal conductivity flows through a tube accompanying with heat transfer between the tube wall and the fluid, the energy balance for laminar flow situation leads to the following differential equation:

$$\rho c_p v_z \frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \frac{\partial^2 T}{\partial z^2} \qquad (1)$$

The initial and boundary conditions are as follows:

For z < 0, the wall of the tube is insulated

For z > 0, either the wall temperature is maintained constant at T_w (for constant wall temperature case) or the heat flux at wall is maintained constant at q (for constant wall heat flux case).

Very far upstream from the entrance to the heated section, the fluid is at uniform temperature T_∞:
As z = 0, for r ≤ R : $T = T_\infty$
If eq(1) is normalized by using the following normalizing variables:

$$r^* = \frac{r}{R},\ T^* = \frac{T - T_\infty}{T_w - T_\infty},\ z^* = \frac{z}{R},\ \text{and}\ v_z^* = \frac{v_z}{v_z}$$

the following equation is obtained:

$$v_z^* \frac{\partial T^*}{\partial \zeta} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right) + \frac{\partial^2 T^*}{\partial z^{*2}} \qquad (2)$$

where

$$\zeta = \frac{z}{R}\frac{k}{\rho c_p R v_\infty} = z^* \frac{\mu}{\rho R v_\infty}\frac{k}{c_p \mu} = \frac{z^*}{N_{Re} N_{Pr}}$$

The boundary conditions become:

At $r^* = 0$, for $\zeta \geq 0$: $\dfrac{\partial T^*}{\partial r^*} = 0$

At $r^* = 1$, for $\zeta > 0$: $T^* = 1$

As $\zeta = 0$, for $r^* \leq 1$: $T^* = 0$

For constant wall heat flux, eq(2) is obtained by normalizing eq(1) using the following normalizing variables:

$$T^* = \frac{T - T_\infty}{qR/k},\ r^* = \frac{r}{R},\ \zeta = \frac{z}{R}\frac{k}{\rho c_p R v_\infty}$$

Here, the boundary conditions are:

At $r^* = 1$, for $\zeta \geq 0$: $\dfrac{\partial T^*}{\partial r^*} = 0$

At $r^* = 1$, for $\zeta > 0$: $\dfrac{\partial T^*}{\partial r^*} = -1$

As $\zeta = 0$, for $r^* \leq 1$: $T^* = 0$

The tern, $\dfrac{\partial^2 T^*}{\partial r^{*2}}$, in eq(2) represents the axial conduction and as mentioned in the literature, the contribution for axial conduction for reasonably high Peclet number can be neglected. In that case, eq(2) for both constant wall temperature and constant wall heat flux becomes:

$$v_z^* \frac{\partial T^*}{\partial \zeta} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial T^*}{\partial r^*}\right) \qquad (3)$$

with same boundary conditions.

EVALUATION OF BULK TEMPERATURE

The expression for normalized bulk temperature is the same for both constant wall temperature and constant wall heat flux situations and can be shown to be equal to

$$T_b^* = \frac{\int_0^1 T^* v_z^* r^* dr^*}{\int_0^1 v_z^* r^* dr^*} \qquad (4)$$

CALCULATION OF LOCAL AND MEAN NUSSELT NUMBER

The expression for local Nusselt number can be shown to be the following by making an energy balance at any cross-section of the tube:

For constant wall temperature:

$$(Nu)_{local} = \frac{-2 \left.\frac{\partial T^*}{\partial r^*}\right|_{r^*=1}}{\left.T^*\right|_{r^*=1} - T_b^*} \qquad (5)$$

For constant wall heat flux:

$$(Nu)_{local} = \frac{2}{T_w^* - T_b^*} \qquad (6)$$

For constant wall temperature the expression for mean or average Nusselt number can be shown to be

$$\langle Nu \rangle = -\frac{\ln T_b^*}{2\zeta},\ \text{where}\ \zeta = \frac{zk}{2R^2 \rho c_p v_\infty}$$

For constant wall heat flux case, analytical integration is not possible because wall temperature is not constant. In this work, the following technique is used:

$$\begin{aligned}\langle Nu \rangle &= \frac{1}{\zeta}\int_0^\zeta (Nu)_{local} d\zeta \\ &= \frac{1}{\zeta}\int_0^\zeta \frac{2}{T_w^* - T_b^*} d\zeta \\ &= \frac{2}{\langle T_w^* - T_b^* \rangle}\ \text{where}\ \langle T_w^* - T_b^* \rangle = \frac{1}{\zeta}\int (T_w^* - T_b^*) d\zeta \\ &= \frac{2\zeta}{\int_0^1 (T_w^* - T_b^*) d\zeta}\end{aligned}$$

METHOD OF SOLUTION

The solution is obtained by using orthogonal collocation method, which provides a mechanism for automatically picking up the collocation points by making use of orthogonal polynomials. This method chooses the trial function y(x) to be the linear combination

$$y(x) = \sum_{i=1}^{N+2} a_i P_{i-1}(x)$$

of a series of orthogonal polynomials $P_m(x)$. The set of polynomials can be written in a condensed form as shown below:

$$P_m(x) = \sum_{j=0}^{m} c_{mj} x^j,\ m = 0,1,\cdots,(N-1)$$

The coefficients c_{mj} are chosen so that the polynomials obey the orthogonality condition

$$\int_a^b w(x)P_k(x)P_m(x)dx = 0,\ k = 0,1,\cdots,(m-1)$$

When $P_m(x)$ is chosen to be the Legendre set of orthogonal polynomials, the weighting function w(x) is unity. Similarly, when the weighting function w(x) is either (1-x) or $(1\text{-}x)^{1/2}$, the function $P_m(x)$ becomes either Jacobi or Chebyschef polynomials. In this work, Jacobi polynomials are used for calculating the roots of orthogonal polynomials.

A detailed discussion on orthogonal collocation is available in APPENDIX, and may also be found in references.

Initially attempt was made to use Gear's routine of adaptive step size control for explicit Runge-Kutta method and then adaptive step size control for polynomial extrapolation using Richardson's extrapolation is used to solve the resulting set of ordinary differential equations but both the methods failed to give solution at ζ=0.001. In both the cases, vary small initial step size (h=0.0001) was chosen but it did not work. Then the package BDFSH is tried and proved to be successful. It is quite capable of giving solutions at $\zeta=10^{-5}$ without any numerical instability. This package is used in this work.

RESULTS AND DISCUSSIONS

Table-1 Value of local Nusselt number for constant wall temperature and fully developed velocity profile

Axial Distance (x/R)/(Re.Pr)	Nusselt No. N=6	Nusselt No. N=9	Nusselt No. N=15	Nusselt No. N=18	Nusselt No. Exact
.000010	64.06975	76.137994	65.619313	62.025642	.000050
46.59100	36.234633	35.956588	35.719506	.000100	34.06408
30.145912	28.316161	28.216870	.000500		16.80843
16.312241	16.271927	16.223096	.001000		13.72820
12.909698	12.837926	12.796726	.002000		10.50650
10.167082	10.143535	10.115072	10.1200	.003000	8.94693
8.864010	8.847568	8.827501	.004000		8.07478
8.056935	8.046610	8.025407	.005000	7.49900	7.484849
7.479132	7.461241	7.4710	.010000	6.03038	6.009648
6.007892	5.997957	6.0020	.020000	4.92627	4.920447
4.920614	4.914215	4.9169	.050000	4.00820	4.008080
4.008614	4.005977	4.0050	.100000	3.71215	3.713388
3.714354	3.726648	3.7101	.200000	3.65724	3.659640
3.660539	3.669354				

Table-1 also shows that the value of local ζ = 0.2 is very close to the asymptotic value of 3.6563.

Table – 1 compares the value of local Nusselt number with analytically obtained values obtained by Robert Lipkis [Lipkis, 1954]. It is quite clear from this table that almost accurate results could be obtained by using 15 collocating points. It collocating points are increased further the accuracy does not increase significantly but requires more computing time. Another important feature is that the Nusselt number decreases very quickly in the entrance section and at the very entrance ($\zeta \le 10^{-5}$), the value of Nusselt number is 4 to 5 times higher than that of at ζ=0.001.

Figure–1 shows radial temperature profile for different axial distance for constant wall temperature boundary conditions. As have been mentioned earlier the gradient is very stiff at the entrance. This stiffness of the gradient at low ζ provides a good test of the numerical techniques. For constant wall temperature, the local Nusselt number and mean Nusselt number are presented in Figure 2 and 3 respectively. These two figure shows that only 3 collocating point is sufficient to give fairly good result in the range of $\zeta > 0.001$. This demonstrates the superiority of Orthogonal Collocation Technique.

For constant wall heat flux problem, the temperature profile, local and mean Nusselt numbers are shown in Figure-4, 5 and 6 respectively. The value of local Nusselt number at ζ = 0.2 is in good agreement with analytical asymptotic value of 48/11=4.36364. Since the evaluation of flow length mean Nusselt number requires numerical integration of local Nusselt number, the value of mean Nusselt number is less accurate than local Nusselt number.

The effect of number of collocating points can be seen from Figure 2 and 3. It is clear from these two figures that for normalized axial distance $\zeta \ge 0.001$, the effect of higher collocating points is insignificant, that is, almost accurate results could be obtained using 3 or 6 collocating points. But for $\zeta < 0.001$, the effect of collocating points is very significant. This same observation is also true for constant wall heat flux problems.

APPENDIX

The method of weighted residuals (MWR) is a general method of obtaining solutions for equations of change. The unknown solution is expanded into a set of trial functions. These functions are specified with adjustable constant, which are chosen to give the beat solution to the differential equation.

For example, to solve a differential equation of the following form:

$$f\left(x, y, \frac{\partial^2 y}{\partial x^2}, \frac{\partial y}{\partial x}\right) = 0$$

A solution may be expressed in the following form:

$$y_n = y_0 + \sum_{i=1}^{N} a_i y_i$$

which satisfies the homogeneous boundary conditions. This trial function is substituted into the differential equation to form the residual

$$R(a_i, x, y) = f(y_0, a_i, y_i)$$

If the trial function were an exact solution, the residual would be zero. In MWR the constants a_i are chosen in such a way that the residual is forced to zero in an average sense. For this, the weighted integrals of the residual are set to zero:

$$\int W_k R(a_i, x, y) dx = 0$$

Finally a criterion or a weighting function, W_k , is chosen. Each MWR procedure is characterized by a different choice of the sequence of N weighting functions W_k in the above equation. For example, the Galerkin method uses $\frac{\partial y_n}{\partial a_i}$ as the weighting functions whereas the Least Square method uses $\frac{\partial R_N}{\partial a_i}$, and the collocation method chooses weighting functions to be the dirac delta $(W_k = \delta(x - x_k))$. When the points for collocation are chosen as some roots of an orthogonal polynomial, the method is known an Orthogonal Collocation.

The governing differential equation requires that the solution to be symmetric about r = 0. Hence it can be expanded in terms of only even powers of r, excluding all the odd powers. These information can be included by constructing orthogonal polynomials that are functions of r^2 as shown below:

$$\begin{aligned} y(r^2) &= \sum_{i=1}^{N} b_i P_{i-1}(r^2) \\ &= \sum_{i=1}^{N} d_i r^{2(i-1)} \end{aligned} \qquad \text{(A)}$$

This polynomial can be forced to be orthogonal by using the following condition:

$$\int_0^1 W(r^2) P_k(r^2) P_m(r^2) r^{(a-1)} dr = 0, \quad k \le m-1 \qquad \text{(B)}$$

where a=1, 2, or 3 for planar, cylindrical or spherical geometry, respectively. Again since the first coefficient of the polynomial is taken as one, the choice of the weighting function $W(r^2)$ completely determines the polynomial, and hence the trial function and the collocating points. For example, let us differentiate eq(A) and take the first derivative and the laplacian of it, where the laplacian is given by

$$\nabla^2 y = \frac{1}{r^{a-1}} \frac{\partial}{\partial r^{a-1}} \left(r^{a-1} \frac{\partial y}{\partial r^{a-1}} \right) \qquad \text{(C)}$$

for three geometries. Thus

$$\begin{aligned} \frac{\partial y}{\partial r} &= \sum_{i=1}^{N+1} d_i (2i-2) r^{2i-3} \\ \nabla^2 y &= \sum_{i=1}^{N+1} d_i (2i-2)[(2i-3) + a - 1] r^{2i-4} \end{aligned} \qquad \text{(D)}$$

Now the collocating points are N interior points in the interval $0 < r_j < 1$ and one boundary point $r_{N+1}=1$. The point r = 0 in not included because the symmetry condition requires that the first derivative be zero at r=0 and that condition is already built into the trial function. Now the derivatives are evaluated at the collocating points to give

$$\begin{aligned} y(r_j) &= \sum_{i=1}^{N+1} d_i r_j^{2(i-1)} \\ \frac{\partial y(r_j)}{\partial r} &= \sum_{i=1}^{N+1} d_i (2i-2) r_j^{2(i-1)} \\ \nabla^2 y(r_j) &= \sum_{i=1}^{N+1} d_i \nabla^2 (r^{2(i-1)}) \Big|_{r_j} \end{aligned} \qquad \text{(E)}$$

In matrix notation, these can be written as:

$$\bar{y} = \bar{\bar{Q}} \bar{d}; \; \overline{\frac{\partial y}{\partial r}} = \bar{\bar{C}} \bar{d}; \; \overline{\nabla^2 y} = \bar{\bar{D}} \bar{d} \qquad \text{(F)}$$

where

$$\begin{aligned} Q_{ji} &= r_j^{2i-2}; \; C_{ji} = (2i-2) r_j^{2i-3}; \text{ and} \\ D_{ji} &= \nabla^2 (r^{2i-2}) \Big|_{r_j} \\ &= (2i-2)^2 r_j^{2i-4}, \text{ for cylindrical geometry} \end{aligned}$$

Solving for $\bar{d}$, one can have the followings:

$$\begin{aligned} \overline{\frac{\partial y}{\partial r}} &= \bar{\bar{C}} \overline{\overline{Q^{-1}}} = \bar{\bar{A}} \bar{y} \\ \overline{\nabla^2 y} &= \bar{\bar{D}} \overline{\overline{Q^{-1}}} = \bar{\bar{B}} \bar{y} \end{aligned} \qquad \text{(G)}$$

Once matrices A and B are calculated, the first and second derivative may be approximated by collocation method. In order to calculate these A and B matrices, only the roots of some orthogonal polynomials are required. In this work, roots of Jacobi polynomials are used which gives minimum errors in the middle regions.

As an illustrative example, if eq(3) is collocated, the resulting set of collocated first order differential equations becomes:

$$v_z^*(r_j)\frac{\partial T_j^*}{\partial \zeta} = \sum_{i=1}^{N+1} B_{ji} T_i^* \qquad \text{(H)}$$

with the following collocated boundary conditions

$$\left.\begin{aligned} &T_j^*(0) = 0, \text{ initialcondition} \\ &T_{N+1}^*(1) = 1 \end{aligned}\right\} \quad \text{For constant wall temperature}$$

and

$$\left.\begin{aligned} &T_j^*(0) = 0, \text{ initial condition} \\ &\sum_{i=1}^{N+1} A_{N+1,i} T_i^* = -1 \end{aligned}\right\} \quad \text{For constant wall heat flux}$$

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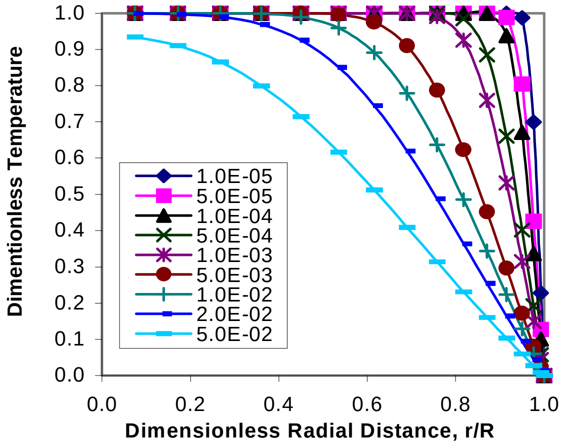

Fig.-1 Radial temperature profile for different axial distance ζ for constant wall temperature. 15 collocating points are used.

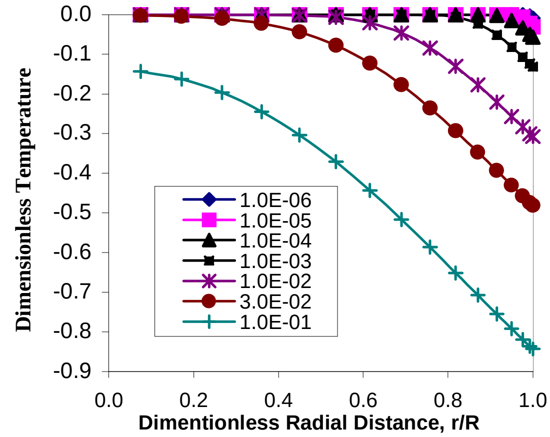

Fig.-4 Radial temperature profile for different axial distance ζ for constant wall heat flux. 15 collocating points are used.

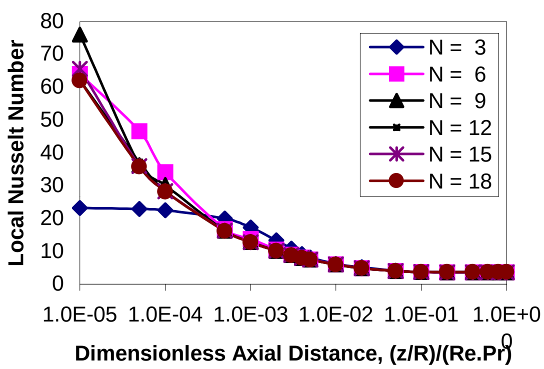

Fig.-2 Variation of local Nusselt number with dimensionless axial distance for constant wall temperature. (N=Number of collocating points)

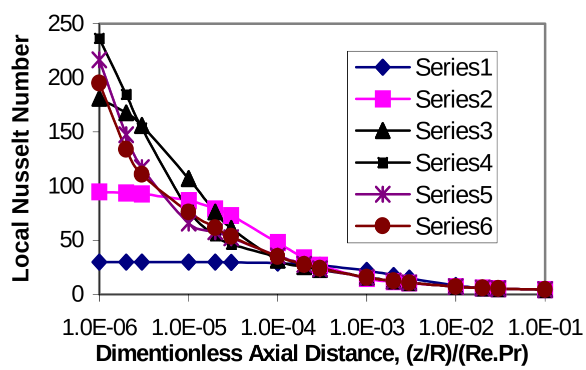

Fig.-5 Variation of local Nusselt number with dimensionless axial distance for constant wall heat flux. (N=Number of collocating points)

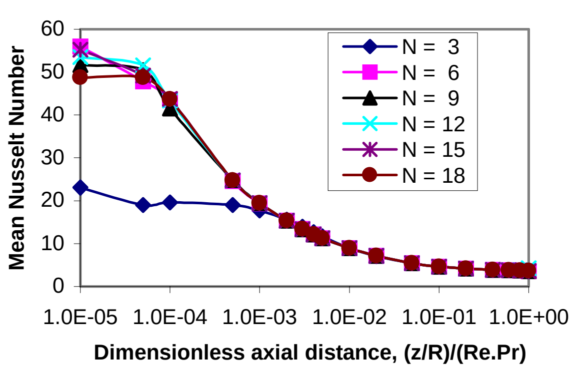

Fig.3 Variation of Mean Nusselt number with dimensionless axial distance for constant wall temperature. (N=Number of collocating points)

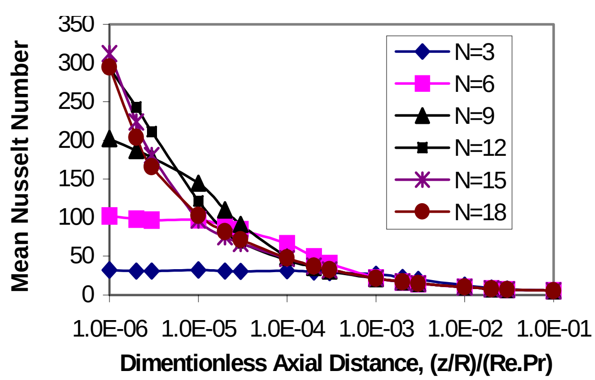

Fig.-6 Variation of Mean Nusselt number with dimensionless axial distance for constant wall heat flux. (N=Number of collocating points)